PA1140 – Waves and Quanta

Unit 3 – Core Exercises Feedback

Exercise 3.1. A photon has a wavelength of 0.2 nm, calculate its energy and momentum.

Energy of photon $E = \frac{hc}{\lambda} \approx \frac{1240}{0.2} = \boxed{6200 \text{ eV} = 0.99\times10^{-15} \text{ J}}$

Momentum of photon $p = \frac{h}{\lambda} \approx \frac{6.63\times10^{-34}}{0.2\times10^{-9}} = \boxed{33\times10^{-25} \text{ Js}}$

Exercise 3.2. The photon from Ex. 3.1 enters a region of plasma where it undergoes Compton-scattering. Assuming each collision deflects the photon by 90 degrees, how many scattering events are necessary to double the wavelength of a photon?

From notes $\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$

So for $\theta = 90°$ *we have* $\Delta\lambda = \frac{h}{mc} \approx \frac{6.63\times10^{-34}}{9.11\times10^{-31}\times3.00\times10^{8}} \approx 2.43 \text{ pm}$

Hence, to go from 200 *to* 400 pm *requires* $\frac{200}{2.43} = 82.3$ *or* $\boxed{83}$ *collisions.*

Exercise 3.3. When light of wavelength λ_1 is incident on the cathode of a photoelectric tube, the maximum kinetic energy of the emitted electrons is 1.8 eV. If the wavelength is reduced to $\lambda_1/2$, the maximum kinetic energy of the emitted electrons is 5.5 eV. Find the work function ϕ of the cathode material.

From notes $KE_{\max} = hf - \phi \qquad \Rightarrow \phi = \dfrac{hc}{\lambda} - KE_{\max}$

From question $\phi = \dfrac{hc}{\lambda_1} - 1.8\text{ eV} = \dfrac{2hc}{\lambda_1} - 5.5\text{ eV}$

$$\Rightarrow \frac{hc}{\lambda_1} = 3.7\text{ eV} \qquad \Rightarrow \boxed{\phi = 1.9\text{ eV}}$$

Exercise 3.4. A 1D wavefunction is given by

$$\psi(x) = A \exp\left(-\frac{x}{a}\right),$$

which is defined in the range $x \geq 0$, where a is a positive constant.

a) Find the normalization constant A in terms of a.

b) Calculate the probability that a particle with this wave-function is found in the range $x < a$.

(a) To normalise set $\int_0^\infty \psi^2 dx = 1$

i.e. $\int_0^\infty A^2 \exp\left(-\frac{2x}{a}\right) dx = 1 \quad \Rightarrow \quad \left[A^2 \exp\left(-\frac{2x}{a}\right)\frac{-a}{2}\right]_0^\infty = 1$

Hence $A^2 = \frac{2}{a} \quad \Rightarrow \quad \boxed{A = \sqrt{\frac{2}{a}}}$

(b) Probability for particle to be in range $\Pr(0 \to a) = \int_0^a \psi^2 dx$

$$\Rightarrow \Pr(0 \to a) = \frac{2}{a}\int_0^a \exp\left(-\frac{2x}{a}\right) dx = \frac{2}{a}\left[\frac{-a}{2} \exp\left(-\frac{2x}{a}\right)\right]_0^a$$

$$= 1 - e^{-2} \quad \approx \boxed{0.865}$$

Exercise 3.5. Electrons are scattered off a virus that is 10^{-8} m in size. Use Heisenberg's Uncertainty Principle to estimate the minimum accelerating voltage required if the electrons are to probe details of the virus down to 1/1000th of its size.

Here we are only interested in a ball-park argument!

To get a resolution of $\sim 10^{-11}$ m *we must effectively localize the electrons to that precision i.e.* $\Delta x \sim 10^{-11}$ m

Thus this corresponds to a momentum uncertainty $\Delta p = \frac{h}{4\pi \Delta x} \approx 5.3\times10^{-24}$ kg m s^{-1}

Electron energy should be at least enough to give this momentum $E > \frac{(\Delta p)^2}{2m} \approx 1.5\times10^{-17}$ J

Finally, convert to eV to get required voltage $\boxed{V > 100 \text{ V}}$

Alternative approach would be to set resolution to de Broglie WL of the electron

$$p = h/10^{-11} \text{ kg m/s} \quad \Rightarrow \quad E > \frac{(p)^2}{2m} \approx 2.5\times10^{-15} \text{ J}$$

Exercise 3.6. Consider wave functions of the form

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \ldots$$

corresponding to a particle in the nth state of a one dimensional box on the x axis in the region of $0 \leq x \leq L$.

Find an expression for the expectation value of x^2 in terms of L and n.

Expectation value $\langle x^2 \rangle = \int_0^L x^2 \psi^2 dx = \int_0^L x^2 \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2}{L} \int_0^L \frac{x^2}{2} \left\{1 - \cos\left(\frac{2n\pi x}{L}\right)\right\} dx$$

Requires integration by parts (twice!)

$$\langle x^2 \rangle = \frac{2}{L}\left[\frac{x^3}{6} - \frac{Lx^2}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) + x\left(\frac{L}{2n\pi}\right)^2 \cos\left(\frac{2n\pi x}{L}\right) + \left(\frac{L}{2n\pi}\right)^3 \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2}{L}\left[\frac{L^3}{6} + x\left(\frac{L}{2n\pi}\right)^2 \cos\left(\frac{2n\pi x}{L}\right)\right]_0^L = \boxed{\frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}}$$