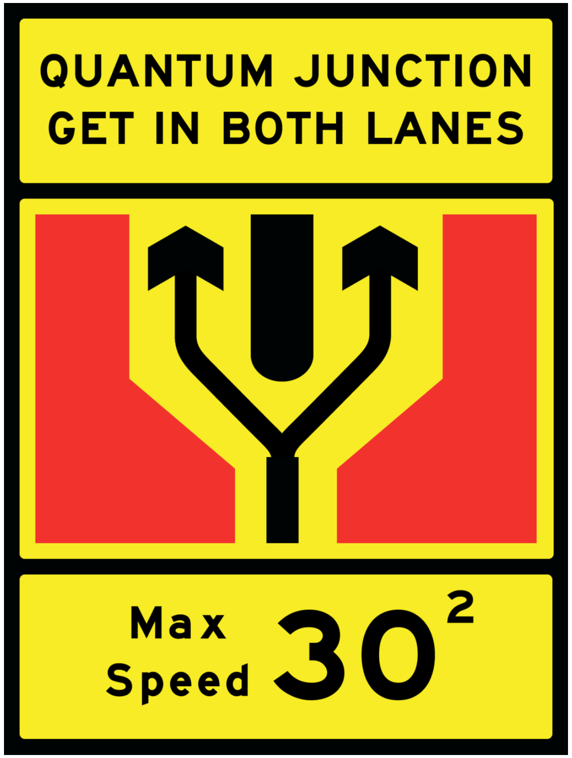
Reminder: http://www.star.le.ac.uk/nrt3/QM/

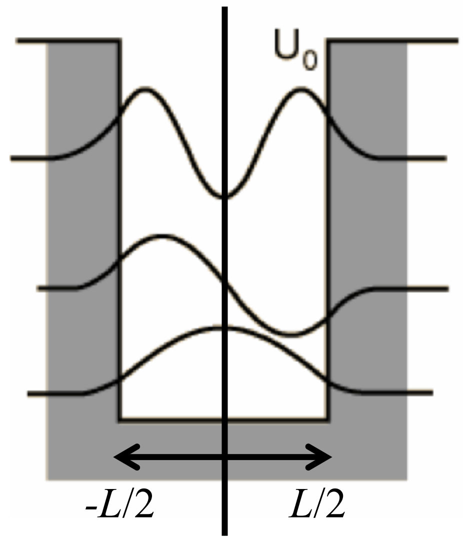
Lecture 3: some more complexity

In this lecture we consider quantum mechanics applied to more complicated potentials.



Finite potential well

If the energy of a state is less than the height of the well, then the solutions look similar to the infinite well. In this case the wave-functions penetrate the walls and decay exponentially. These solutions can be found by solving the Schrodinger equation in each region (take origin at centre of well!).



<u>Outside well</u>

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi(x)$$

$$\Rightarrow \psi(x) = Ce^{+\alpha(x+L/2)},\ \psi(x) = De^{-\alpha(x-L/2)}$$

where:

$$\alpha^2 = \frac{2m}{\hbar^2}(U_0 - E)$$

<u>Inside well</u>

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}$$

$$\Rightarrow \psi(x) = A\sin kx + B\cos kx$$

where:

$$k^2 = \frac{2mE}{\hbar^2}$$

Finite potential well

Fix all the constant coefficients by normalising and enforcing **continuity** and **differentiability** at the well boundaries. For example, consider just the symmetrical (odd n) cases.

Symmetry implies $C = D$ and $A = 0$.

$$\psi(L/2)_- = \psi(L/2)_+ \quad \Rightarrow \quad B\cos(kL/2) = C$$

$$\psi'(L/2)_- = \psi'(L/2)_+ \quad \Rightarrow \quad -Bk\sin(kL/2) = -\alpha C$$

Dividing gives: $\alpha = k\tan(kL/2)$

$$\Rightarrow \quad U_0 - E = E\tan^2\left(L\sqrt{\frac{mE}{2\hbar^2}}\right)$$

Hence allowed (quantised) energy levels are those which obey this constraint.

The Harmonic Oscillator

Another relatively simple system, which approximates many real world situations, is the harmonic oscillator, in which the potential is parabolic (cf. mass on spring). Thus the Schrodinger equation becomes:

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{m\omega^2}{2}x^2\psi(x)$$

In this situation the energy levels are simply separated by a constant difference. Consider a (ground-state) solution of a Gaussian form:

$$\psi(x) = Ce^{-ax^2/2}$$

Differentiating twice and putting back in the Schrodinger equation we find:

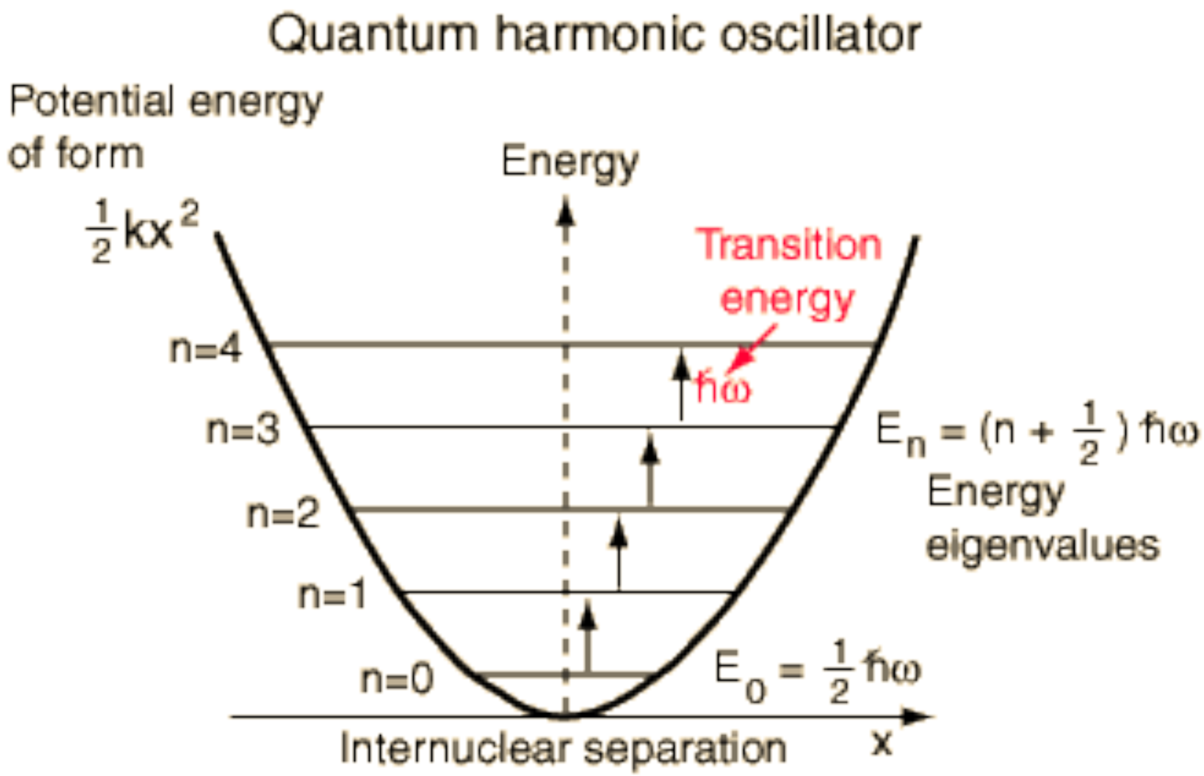
$$E\psi(x) = -\frac{\hbar^2}{2m}(a^2x^2 - a)\psi(x) + \frac{m\omega^2}{2}x^2\psi(x)$$

Thus we must have:

$$a = \frac{m\omega}{\hbar} \quad ; \quad E_0 = \frac{\hbar\omega}{2}$$

The Harmonic Oscillator

It turns out that the (more complicated) higher energy solutions have regularly spaced energy levels:



The Harmonic Oscillator

For example, the first excited state is:

$$\psi(x) = Dxe^{-ax^2/2}$$

Differentiating twice and putting back in the Schrodinger equation we find:

$$E\psi(x) = -\frac{\hbar^2}{2m}(a^2x^2 - 3a)\psi(x) + \frac{m\omega^2}{2}x^2\psi(x)$$

Thus, as claimed:

$$E_1 = \frac{3\hbar\omega}{2}$$

Example

A particular potential produces a wavefunction of the following form:

$$\psi(x) = k(x - x^2); \quad 0 < x < 1, \quad 0 \text{ elsewhere}$$

Find expectation value for position.

Steps: first normalise to find k, then find expectation.