University of Leicester
Department of Physics and Astronomy
Lecture Notes
2nd Year Optics

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Contents

1 BOOKS

- Optics, E.Hecht
- Insight into Optics, O.S.Heavens and R.W.Ditchburn
- Fundamentals of Optics, F.A.Jenkins and H.E.White, 4th Edition

2 INTRODUCTION

Optics is a very large and complicated subject covering all aspects of the study of light. It is usually divided up under 3 main headings:

- Geometric Optics - light is something which travels in straight lines or rays. It has a speed and a path. This leads to a description of reflection, refraction and dispersion. The approach is purely kinematic - a description of motion.
- Wave Optics - light is a travelling wave characterised by v,λ and ν. This leads to the phenomena of interference, diffraction, polarization, double refraction. It is closely connected to the EM theory which you do in the other two thirds of this course. Wave optics and classical EM theory are essentially the dynamics of light - motion with reference to forces.
- Quantum Optics - light is composed of particles or photons. This leads to the processes of emission and absorption. It is required for the study of atomic spectra, lasers and scattering. This embodies quantum mechanics which seeks to find fundamental mechanisms for phenomena. Specifically in optics mechanisms of emission and absorption of light.

In this course we will consider the first 2 leaving quantum optics for the 3rd/4th year. We will look at the same basic phenomena as the EM theory but from a different viewpoint. You should remember it is the same stuff that we are talking about whether we call it light or an EM wave.

Going back to the fundamental EM equations is often tedious and unnecessary. Optics provides us with a different set of tools with which to solve problems.

3 THE SPEED OF LIGHT

I.Newton (1642-1727) - corpuscular theory of light predicted it should be faster in matter.

C.Huygens (1629-95) - wave theory predicted it should be slower in matter.

It was first measured in air in 1849 by A.H.L.Fizeau. He used a toothed wheel to chop the light into bursts or pulses. The pulses were reflected off a distant mirror to give a path length of 17.3 km. The rotation speed of the wheel was adjusted to block or transmit the returning pulses. A transit time of 5.5×10^{-5} seconds gave $c = 3.15 \times 10^{8}\ m\ s^{-1}$.

It was found to be slower in water than in air by J.B.L.Foucault in 1850.

James Clerk Maxwell (1831-79) provided the foundations of the EM theory predicting the speed in vacuum to be $c = 1/\sqrt{\varepsilon_o \mu_o}$.

Henrich R.Hertz (1857-94) generated and detected radio waves in 1888 verifying the existence of long wavelength EM radiation.

In 1879, 1882 and 1926 A.A.Michelson made successively more accurate measurements of c thus verifying the Maxwell theory and linking light with other forms of EM radiation.

Although by terrestial standards the speed of light is high in the context of the larger Universe it is modest!

1 A.U. (the radius of the orbit of Earth around the sun) $= 1.5 \times 10^{13}\ cm \equiv 8.3$ light minutes.

α Centauri, the nearest star is 5 light years away $\equiv 4.7 \times 10^{18}\ cm$

4 REFRACTIVE INDEX

$n = \frac{speed_{vacuum}}{speed_{medium}} = \frac{c}{v}$

It is a quantity that can be measured.

$n_{glass} = 1.52$

$n_{water} = 1.33$

$n_{air} = 1.000292$ at S.T.P.

5 FERMAT'S PRINCIPLE - OPTICAL PATH LENGTH

Pierre de Fermat 1608-1665

Consider the product $vt = d$

$n = c/v$ so $d = ct/n$ or $nd = ct = \Delta$ the *optical path length.*

The total optical path length can be calculated over several media using a sum $\Delta = n_1 d_1 + n_2 d_2 + \cdots$ or using an integral if the refractive index varies continuously.

$$\Delta = \int_A^B n(\underline{r})\underline{dl}$$

where A and B are the end points of the path.

Fermat's Principle can be written in a number of ways:

- The optical path length along the actual path of a ray of light is the same as that for closely adjacent paths.
- A light ray takes a path which is stationary with respect to the optical path length.
- Light rays take the path of shortest time.

Note that the last statment above was the original form of Fermat's Principle but with the advent of relativity we must now be careful with this formulation.

If n is constant then the actual path of a ray will be a straight line so Fermat's Principle applied in this case says:

Light travels in straight lines - rectilinear propogation!

Note that equal optical paths contain an equal number of wavelengths for light of the same frequency. Remember this when we do Huygen's Principle later.

6 REFLECTION

We can use Fermat's Principle to determine the direction of a reflected ray from a plane mirror. See figure 1.

If points A and B are at distances h and h' from the mirror we can calculate the optical path length for a ray that hits the mirror at a distance x from the point below A and a distance $a - x$ from below B where $a = constant$. If the refective index is n:

$\Delta = n(\sqrt{h^2 + x^2} + \sqrt{h'^2 + (a - x)^2})$

Differentiating with respect to the position x we get:

$$\frac{d\Delta}{dx} = nx(h^2 + x^2)^{-1/2} + n(x - a)(h'^2 + a^2 + x^2 - 2xa)^{-1/2}$$

For a minimum (stationary value):

$$\frac{x}{\sqrt{h^2 + x^2}} = \frac{a - x}{\sqrt{h'^2 + (a - x)^2}}$$

So $\sin\theta_i = \sin\theta_r$ or $\theta_i = \theta_r$.

In a more complicated situation Δ can be a function of more than 1 variable and is described by a surface. In such a case we must search for the minimum of $\Delta(x, y, z \ldots)$ over the surface.

7 REFRACTION

We can us Fermat's Principle to deduce the law of refraction at a plane interface. See figure 2. The result is *Snell's Law*:

$n_1 \sin\theta_1 = n_2 \sin\theta_2$

Make sure you can derive this using Fermat's Principle.

Snell deduced his law from experimental data and thought that cosecants fitted his data best. The frenchman Descarte was the first to use the correct sines. In France it is known as Descarte's Law!

Figure 3 illustrates the consequences of multiple refractions, *bending* of light.

8 REVERSIBILITY

In Fermat's Principle the optical path length can be calculated in either direction, A to B or B to A, the result is the same. This leads to the principle of *geometrical reversibility.*

Note that the symmetry in Snell's Law, you can swap the subscripts without changing the equation. This is a consequence of reversibility. We have symmetry in time.

9 DISPERSION

Experimentally we find that the refractive index n depends on the colour of the light. We can use refraction to *split* white light into component colours.

Standard emission lines are used to designate the colour following the original nomenclature of Fraunhofer for prominent lines in the Solar spectrum.

Normal dispersion results in decreasing n moving to the red end of the spectrum (longer wavelengths). See figure 4.

The normal dispersion of glass is given by:

$n_F = 1.52933$ (486.1 nm)

$n_D = 1.52306$ (589.0 nm)

$n_C = 1.52042$ (656.3 nm)

Anomalous dispersion results from absorption structure. See figure 5. In this case there are discontinuities in n across the spectrum caused by absorption bands peculiar to the medium. The refractive index is seen to increase or jump up as the wavelength increases, hence anomalous. All substances exhibit anomalous dispersion over certain wavebands.

The dispersive power is defined as:

$$V = \frac{n_F - n_C}{n_D - 1}$$

10 THE STOKES TREATMENT OF REFLECTION AND REFRACTION

Consider 3 rays of light at a plane interface as illustrated in figure 6. If there is NO absorption then we can use the principle of reversibility to deduce relationships between the amplitudes of the 3 rays. We assume that the sum of the amplitudes is independent of direction at the interface.

We use primed quantities to represent reflection and transmission from the dense to rare medium, unprimed for rare to dense.

Comparing the two ways of looking at the same situation we have:

$r^2 + t't = 1$

$tr' + rt = 0$

Rearranging gives:

$tt' = 1 - r^2$

$r = -r'$

These two relationships between the amplitude reflection and transmission coefficients at an interface will be useful in the latter part of the course.

Remember in practice r,r',t and t' are functions of the angle of incidence and they act on the amplitudes.

The negative sign represents a phase difference of π. If there is no absorption then there is a phase difference of π between internal and external reflections.

11 THE FRESNEL EQUATIONS

The coefficients r and t above are properly described by the Fresnel Equations which you do in the EM part of the course. We will return to these when we consider polarization. Note that these equations take into account the effects of absorption using a complex refractive index.

At normal incidence the reflected intensity (power) fraction is given by:

$R = r^2 = (\frac{n_2 - n_1}{n_2 + n_1})^2$

For glass $n_2 \approx 1.5$ and $R \approx 0.04$; 4% of the optical light is reflected.

12 HARMONIC WAVES

Behaviour like dispersion, the Stokes treatment of reflection and refraction (and interference-diffraction to come) suggest that light has wave like characteristics.

A harmonic wave in 1 dimension has the form:

$\psi = A\cos(\omega t - kx + \phi)$ or $\psi = A\exp i(\omega t - kx + \phi)$

k is the wave number and ω is the angular frequency.

$k = 2\pi/\lambda$, $\omega = 2\pi\nu$ and the velocity $v = \lambda\nu = \omega/k$.

Unlike a classical particle a wave has extent. It spreads out over space. Mathematically it is described by a *field*. In 3 dimensions we use the complex form:

$\psi = A\exp i(\omega t - \underline{k}.\underline{r} + \phi)$

Wavefronts are surfaces of constant phase within the space occupied by the wave.

$\underline{k}$ is a *wave vector* with amplitude $2\pi/\lambda$ and direction perpendicular to the wavefronts.

Imagine you are at some point $\underline{r}_o$

If some position vector $\underline{r} = \underline{r}_o + \underline{a}$ then $\underline{k}.\underline{r} = \underline{k}.\underline{r}_o + \underline{k}.\underline{a}$.

If $\underline{a}$ is parallel to the wavefronts at $\underline{r}_o$ then $\underline{k}.\underline{r}$ is constant (constant phase).

If $\underline{a}$ is perpendicular to wavefronts then it is pointing the direction of the wave and $\underline{k}.\underline{a}$ is the change in phase with respect to $\underline{r}_o$.

13 ADDITION OF HARMONIC WAVES OF THE SAME FREQUENCY

$\psi_1 = a_1 \cos(\omega t + \phi_1)$

$\psi_2 = a_2 \cos(\omega t + \phi_2)$

$\psi_1 + \psi_2 = \psi = A \cos(\omega t + \phi)$

The resultant is a harmonic wave of the same frequency with:

$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_2 - \phi_1)$$

$$\tan\phi = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}$$

If $a_1 = a_2 = a$ and $\phi_1 - \phi_2 = \delta$ then:

$$A^2 = 2a^2(1 + \cos\delta) = 4a^2 \cos^2(\delta/2)$$

If $\delta = 0$ then A is a maximum and we get *constructive interference*.

If $\delta = \pm\pi$ then A is a minimum and we get *destructive interference*.

As you will see δ often depends on the optical path length of the two light waves. The above formulae are used a lot.

14 PHASOR DIAGRAMS

In an Argand diagram the complex wave components can be represented by vectors. The zero phase or reference axis rotates at angular velocity ω. The projection of a phasor onto the real axis represents the real wave, a measurable amplitude.

Addition of the waves is accomplished by vector addition. See figure 7.

Such phasor diagrams are used a great deal in optics as you will see.

15 ADDITION OF HARMONIC WAVES OF DIFFERENT FREQUENCY

$\psi_1 = a_1 \cos(\omega_1 t + \phi_1)$

$\psi_2 = a_2 \cos(\omega_2 t + \phi_2)$

If $a_1 = a_2 = A$ then

$$\psi_1 + \psi_2 = \psi = A \cos \frac{1}{2}(\omega_1 t - \omega_2 t + \phi_1 - \phi_2) \cos \frac{1}{2}(\omega_1 t + \omega_2 t + \phi_1 + \phi_2)$$

The result is the product of 2 components, a harmonic wave with a frequency equal to the mean frequency $\frac{1}{2}(\omega_1 + \omega_2)$ and an envelope or beat with a frequency equal to half the difference frequency $\frac{1}{2}(\omega_1 - \omega_2)$. See figure 8.

16 FOURIER SYNTHESIS

The general extension of the above addition to many discrete frequencies is covered by the Fourier Theorem.

The Fourier Theorem - any periodic function with period λ can be expanded as a summation of harmonic terms:

$$f(x) = a_0/2 + \sum_{m=1}^{\infty} (a_m \cos mkx + b_m \sin mkx)$$

where $k = 2\pi/\lambda$.

In the limit when there is a continuous spectrum of frequencies we use Fourier Integrals.

The Fourier Integrals - if we let $\lambda \to \infty$ we get an aperiodic function and the summation becomes an integral.

$$f(x) = 1/(2\pi) \int_{-\infty}^{+\infty} F(k) \exp(ikx) dk$$

$$F(k) = \int_{-\infty}^{+\infty} f(x) \exp(-ikx) dx$$

This is often expressed using the notation $f(x) \Leftarrow FT \Rightarrow F(k)$

The functions $f(x)$ and $F(k)$ are called a Fourier Transform pair.

The Fourier transform of the top hat function is shown in figure 9a. The Fourier transform of a truncated cosine was is shown in figure 9b. Figures 9c - 9e illustrate the Fourier decomposition of a top hat function and the way in which successively adding cosine terms converges to the final answer.

Suppose we have a pulse or wave packet which is a truncated harmonic wave with a start and an end.

$f(x) = a \cos k_p x$ for $-L \leq x \leq L$ and $f(x) = 0$ elsewhere. See figure 9.

We can calculate the Fourier Transform by performing the integration:

$$F(k) = aL(\frac{\sin(k_p + k)L}{(k_p + k)L} + \frac{\sin(k_p - k)L}{(k_p - k)L})$$

If $k_p \gg 2\pi$ then the number of periods in $2L$ is large and the first term is small so:

$$F(k) = aL\frac{\sin(k_p - k)L}{(k_p - k)L}$$

The positions of the first minima are at $k_p \pm \pi/L$. The packet is described by the *integral* (summation) of a band of harmonic components centred on k_p.

$\Delta k = 2\pi/L$ and as $L \to \infty$ so $\Delta k \to 0$.

$F(k)$ is said to represent the spectrum of the wavepacket.

17 PHASE VELOCITY - GROUP VELOCITY - DISPERSION

ALL physical manifestations of travelling harmonic waves involve wavepackets or groups because of the finite extent of space and/or time.

There are 2 velocities associated with wavepackets or groups.

Harmonic waves or harmonic components have a *phase velocity* $v = \omega/k$. This is the velocity at which the wavefronts travel.

If $\phi = A\cos(\omega t - kx)$ then wavefronts are given by $\omega t - kx = 2\pi m$ where m is an integer. So $x = (\omega t - 2\pi m)/k$ and $dx/dt = \omega/k$ QED.

A group of harmonic waves or packet has a *group velocity* $v_g = d\omega/dk$. This is the velocity at which the packet shape or envelope travels. It is the velocity at which *information* or *energy* is transported.

If the envelope has the form $\phi = \cos(\omega_1 t - \omega_2 t - k_1 x + k_2 x)$ then stationary values in the envelope pattern are given by $(\omega_1 - \omega_2)t - (k_1 - k_2)x = 2\pi m$ where m is an integer. So $dx/dt = (\omega_1 - \omega_2)/(k_1 - k_2) = \Delta\omega/\Delta k$ QED.

Since $\omega = vk$ we have $v_g = d\omega/dk = v + kdv/dk$

If v is independent of k then $v_g = v$ and the medium is non-dispersive, $n = constant$.

But $n = c/v = kc/\omega$. Note n is defined using the phase velocity. Substituting for v gives:

$v_g = (c/n)(1 - (k/n)dn/dk)$

If $dn/dk > 0$, $v_g < v$ and we have *normal dispersion*.

If $dn/dk < 0$, $v_g > v$ and we have *anomalous dispersion*.

In 1885 A.A.Michelson measured the group velocity of light in carbon disulphide by using pulses of light. He found:

$n_g = c/v_g = 1.758$ compared with $n = 1.1635$

If there is dispersion then the shape of a wave packet changes as it propogates because the different harmonic components travel at different speeds and therefore change phase relative to one another. In general a pulse *spreads out* as it propogates.

An easy way to *see* the group and phase velocity in action is in the pattern of ripples spreading out from a pebble thrown into a still pond. The ripples form a wavepacket and the dispersion of such water ripples is such that $v_g \approx v/2$. So individual ripples move through the *packet*. They appear to be created at the trailing edge of the pattern, travel through the pattern and are destroyed at the leading edge. Groups of waves beloved by surfers behave in the same way as they spread out from the large ocean storms which create them.

18 LIGHT AND THE UNCERTAINTY PRINCIPLE

You were introduced to The Heisenburg Uncertainty Principle in the first year course on Physics of the Atom. A particle has a position and momentum but if we try to measure them both there is always an uncertainty expressed by:

$\Delta x \Delta p \geq h/2\pi$

This is a consequence of the wave nature of particles.

The de Broglie relationship tells us that the momentum of a particle is related to its wavelength:

$p = h/\lambda$

If we pass light through a small slit of width d then we know that in the direction of the slit light particles (photons) are confined to $\Delta x \approx d$. Therefore there is an uncertainty in the momentum of the photons in the direction of the slit given by the equation above. Consequently we are uncertain about which direction the photons are travelling after they leave the slit:

$\Delta\theta \approx \Delta v/v = \Delta p/p$

Substituting from above we get:

$\Delta\theta \sim \lambda/(2\pi d)$

The light is said to be diffracted by the slit and $\Delta\theta$ is called the diffraction angle. Diffraction comes about because light has the characteristics of a wave. In the next few lectures we will look at this in much more detail.

19 THE SCALE OF LIGHT WAVES

It is important that you appreciate the scale of light waves. A typical wavelength in the middle of the visible band is $5000\AA$. So a laser beam which is 1 mm across is $\sim$ 200 wavelengths across.

Compare this with sea waves of wavelength 10 m. The cross-section of the laser beam is equivalent to 2 km of shore line.

When we draw diagrams of wavefronts and obstacles etc. you should bear this analogy in mind.

Individual atoms in a solid are separated by $\approx 1\AA$. Each atom in a glass through which visible light is travelling is rather like a 2 mm cork bead in the face of seawaves.

As far as visible light is concerned matter (even gas at reasonable pressure) is quasi continuous.

20 HUYGENS' PRINCIPLE - WAVEFRONTS AND WAVELETS

Surfaces in space over which a travelling wave disturbance has a constant phase are called *wavefronts*.

As such wavefronts move through space their speed is determined by the phase velocity - usually expressed as a refractive index in optics.

If we consider 1 wave of frequency ν, as n changes, so λ must change. Hence the separation of successive wavefronts changes as a function of n.

Huygens' Principle gives us a way to construct successive wavefronts geometrically. It states that:

Every point on a wavefront acts as a source of *spherical secondary wavelets* such that after some time (Δt) the primary wavefront lies on the envelope defined by the secondary wavelets. The radius of the secondary wavelets will be $v\Delta t$. See figure 10.

In n varies with position we must take $\Delta t \rightarrow dt$ to construct the new wavefronts. If n is constant we can take larger steps.

Huygens imagined that each wavelet was generated by the vibration of some elastic medium or aether. Although this is not the case the geometric construction still tells us how the wavefronts will propogate.

Note that Huygens' Principle says nothing about the amplitude of the wavefronts or wavelets.

21 REFLECTION AND REFRACTION OF WAVEFRONTS

Suppose we have a monochromatic parallel beam incident at a plane surface. The wavefronts in the beam are a set of parallel planes perpendicular to the propogation direction. We can use the wavelet construction to find the position of the reflected and refracted wavefronts. Since wavelets in the reflected beam are travelling at the same speed as those in the incident beam the law of reflection holds $\theta_r = \theta_i$. However the refracted wavelets travelling into the denser medium, refractive index n_2, will be retarded compared with the reflected beam. See figure 11. Since the wavefronts are planar we immediately see:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

But since $v_1/v_2 = n_2/n_1$ we have Snell's law:

$n_1 \sin\theta_1 = n_2 \sin\theta_2$

22 FERMAT VERSUS HUYGENS

Fresnel showed that Huygens' Principle is a direct result of solving the wave equation within a medium. Huygens' wavelets correctly describe the propogation of a progressive harmonic wave.

$\psi(\underline{r}, t) = A \exp i(\omega t - \underline{k}.\underline{r})$

where the amplitude of $\underline{k}$ is given by

$|\underline{k}| = n\omega/c$

As the refractive index changes so the wavevector (wavelength) changes. $\underline{k}$ defines the direction of travel which is perpendicular to the envelope of the wavelets. See earlier discussion of moving along a wavefront.

Fermat's Principle considers the direction of $\underline{k}$ at a point whereas Huygens' Principle considers the development of wavefronts.

Both principles lead to the same laws of reflection and refraction. Both describe aspects of the behaviour of a progressive wave.

Neither principle tells us anything about the physics of the wave. There is no mention of amplitudes, electric or magnetic fields, emission, absorption, polarization, just propogation.

23 REFRACTION AT SPHERICAL SURFACES

Consider a spherical interface between 2 media, refractive index n_1 on the left and $n_2 (> n_1)$ on the right. C is the centre of curvature. We shall use Fermat's Principle and consider a ray that starts at S, meets the interface at A, is refracted and ends up at P. This is illustrated in figure 12.

The optical path length is:

$\Delta = n_1 l_1 + n_2 l_2$

We can use the cosine rule on triangles SAC and ACP to find l_1 and l_2

$l_1 = (R^2 + (s_1 + R)^2 - 2R(s_1 + R)\cos\phi)^{1/2}$

$l_2 = (R^2 + (s_2 - R)^2 - 2R(s_2 - R)\cos\phi)^{1/2}$

Now we find the condition for a stationary path wrt ϕ. That is moving A around. Differentiating wrt ϕ we get:

$$\frac{n_1 R(s_1 + R)\sin\phi}{2l_1} - \frac{n_2 R(s_2 - R)\sin\phi}{2l_2} = 0$$

or

$$\frac{n_1}{l_1} + \frac{n_2}{l_2} = \frac{1}{R}\left(\frac{n_2 s_2}{l_2} - \frac{n_1 s_1}{l_1}\right)$$

If we consider ϕ small, h small, then $l_1 \approx s_1$ and $l_2 \approx s_2$ so:

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}$$

We could have used Snell's law at point A and approximated $\sin\phi \approx \phi$ for small angles, the result would have been the same.

Rays for which h is small are known as *paraxial rays* and the optical formulae derived under this approximation form what is known as paraxial, first order or Gaussian optics since it was Gauss who first exploited the approximation in 1841.

Note the spherical surface converts diverging spherical wavefronts into converging spherical wavefronts under the paraxial approximation.

24 THIN LENSES

We can now combine 2 spherical surfaces into a *thin lens.* Each surface refracts spherical wavefronts in accordance with the above formula.

$n_m/s_1 + n_l/s_2 = (n_l - n_m)/R_1$ for first surface

$n_l/s_3 + n_m/s_4 = (n_m - n_l)/R_2$ for second surface

If the lens is thin, $d \ll$ s's, then $s_2 = -s_3$. Note the sign convention: a negative sign represents a virtual object, the wavefronts are converging towards second surface.

Combining the 2 formulae gives:

$n_m/s_1 + n_m/s_4 = (n_l - n_m)(1/R_1 - 1/R_2)$

where s_1 is the object distance from the thin lens and s_4 is the image distance.

We define the *primary focus* of the lens to be s_1 when $s_4 \to \infty$:

$n/f = (n_l - n_m)(1/R_1 - 1/R_2)$

Usually $n_m = 1$ and we let $s_1 = u$ and $s_4 = v$ so:

$1/u + 1/v = 1/f$

This is called the Gaussian lens formula. We also have:

$D = 1/f = (n_l - 1)(1/R_1 - 1/R_2)$ the lens maker's formula.

D is known as the *power* of the lens. Units of diopters, (m^{-1}).

For thin lenses in contact it is easy to show that:

$D_{12} = D_1 + D_2$

The power of a single surface is $\pm(n_l - 1)/R$ where a positive sign represents converging and negative diverging.

The following self consistent sign convention is normally used when light is travelling from left to right:

positive sign for: real object to left, real image to right

negative sign for: virtual object to left, virtual image to right

focal length positive for converging, negative for diverging

object height is positive when up, negative when down

25 SPHERICAL AND PARABOLIC MIRRORS

A spherical mirror can also be used to focus light as shown in figure 13. See if you can use Fermat's Principle to deduce the *mirror formula* for a spherical mirror.

$$\frac{1}{s_1} + \frac{1}{s_2} = -\frac{2}{R} = \frac{1}{f}$$

where R is the radius of curvature (> 0 if the mirror is convex) and s_1 and s_2 are the object and image distances. This formula only holds for paraxial rays. However if the mirror is parabolic and $s_1 \to \infty$ then ALL rays will be brought to a focus. What is the equation of the parabola if the focus is at the origin?

26 TWO SOURCE INTERFERENCE

When a travelling wave passes through an aperture or is obstructed by an obstacle then there is always some spreading of the wave into regions which are not directly exposed to the original wavevector. Using Huygens' construction wavelets at the edges produce an envelope which is at an angle to the original wavefront. This phenomenon is known as *diffraction*.

So using a narrow slit plane wavefronts can be converted to cylindrical wavefronts.

When two sets of such wavefronts overlap we get two source *interference*. In 1801 Thomas Young demonstrated that two light sources could be made to *interfere*. See figures 14 and 15.

To the right of the slits we have a region where two sets of spherical wavefronts overlap. What intensity distribution do we see on the screen?

The experimental arrangement is such that the wavefronts arriving at the slits are in phase. Consider a point on the screen P.

The *path difference* between the wavefronts from S_1 and S_2 is:

$\Delta = S_2P - S_1P$

The *phase difference* is given by:

$\delta = \Delta 2\pi/\lambda$

Where λ is the wavelength. Note if the refractive index were n around the slits than we could get the same result using the difference in the optical path length and the wavelength in a vacuum.

Therefore at P we have two waves of the same frequency but with phase difference δ. We add the amplitudes to find the resultant amplitude and square to find the intensity. See earlier for the formula for the addition of two waves of the same frequency.

Assuming both waves have the same amplitude a, which will be a good approximation if the angles are small and the slits are the same size, the intensity at P is:

$I = A^2 = 4a^2 \cos^2(\delta/2)$

This is illustrated in figure 16. If the distance to the screen s is much greater than the separation of the slits d and we restrict ourselves to small angles θ then:

$\Delta = d \sin\theta \approx dy/s$

where y is the position of P on the screen.

We get constructive interference if $\delta = 0, 2\pi, \ldots$

Therefore if $dy_m/s = 0, \lambda, 2\lambda, \ldots = m\lambda$ we get a bright fringe.

We get destructive interference if $\delta = \pi, 3\pi, \ldots$

Therefore if $dy_m/s = \lambda/2, 3\lambda/2, \ldots = m\lambda$ we get a dark fringe.

The pattern seen is cosinusoidal in intensity. Note that the average intensity is the same as expected if no interference were taking place. We have no loss or gain of energy just a redistribution.

27 WAVEFRONT SPLITTING

Young's slits is an example of a wavefront splitting interference. There are several practical arrangements that can be used to achieve the same effect. These are illustrated in figures 17a,b and c.

- Young's slits
- Fresnel's double mirror
- Fresnel's biprism
- Lloyd's mirror

An extra complication is introduced in the case of Lloyd's mirror. Only one set of wavefronts suffers a reflection. This introduces a phase change which is $\approx \pi$ so at the join between the screen and mirror planes the waves are out-of-phase. Hence we get a dark fringe along this line rather than a bright fringe. The pattern is shifted by half a fringe.

In all the above arrangements the fringes are *real* and *non-localised.* They exist in a region of space where the wavefronts overlap. No lens is required to create them.

In all cases the light is monochromatic.

Note that in all the arrangements the original source of the wavefronts is the same. Interference NEVER occurs unless this is the case.

We can't describe interference-diffraction using Fermat's Principle, we must use Huygen's Principle.

28 AMPLITUDE SPLITTING - THE MICHELSON INTERFEROMETER

The amplitude of wavefronts in a beam can be *split* into component beams using a semi-transparent plate or *beam splitter* set at an angle to the source wavefronts.

In the Michelson interferometer the two beams reflect off two mirrors and are recombined by the beam splitter as illustrated in figure 18.

To see fringes the combined beams can be viewed with a telescope or the eye focused at infinity. It is also possible to see real fringes by placing a *screen* near the exit aperture.

Note that the source is diffuse and monochromatic. Each source point is producing wavefronts. To find out what fringe pattern is expected we unfold the interferometer so that the arms are parallel and consider just 1 source point. The geometry is shown in figure 19.

Two images are seen at a separation $2d$ where d is the difference in the arm lengths (optical path lengths).

$\Delta = 2d \cos\theta$

One beam suffers an internal reflection while the other suffers an external reflection so there is also a relative shift of π (see Stokes treatment earlier). Consequently we get destructive interference when:

$2d \cos\theta_m = m\lambda$ where m is an integer (the order).

The observer sees alternate light and dark rings corresponding to successive value of m. To see a large number of fringes you require a large telescope aperture.

The order m is a maximum for θ_m small or zero corresponding to the centre of the pattern.

If $d = 0$ then the entire field is filled by 1 dark fringe. This is called the *point of coincidence*. It is independent of wavelength.

The fringes seen in this way are called *fringes of equal inclination* or *Haidinger fringes* after the Austrian physicist. Such fringes are formed by focusing parallel rays (plane wavefronts).

If the brightness of the images seen in the 2 arms is equal then the intensity (amplitude squared) is given by:

$I = 4a^2 \cos^2(\delta/2)$ where

$\delta = 2d \cos\theta(2\pi/\lambda) + \pi$

So to move across the fringes we can:

- fix d and vary θ - circular fringe seen by eye
- fix $\theta(= 0)$ and vary d - move 1 of the mirrors
- fix d and θ and change λ - change n in 1 arm

The Michelson Interferometer is used for:

- Measuring optical surfaces - plane, spherical, aspherical
- Measuring distance accurately by counting fringes
- Measuring refractive index - optical path
- Search for the luminiferous ether

29 DIELECTRIC FILMS

Consider a transparent parallel sided film of dielectric thickness d, illuminated by a monochromatic source at some distance. Plane wavefronts will hit the film at θ_i.

Some of the light will be reflected and some will be transmitted into the film. The refracted wavefronts meet the other side of the film and again a fraction is reflected while the remainder is transmitted. Finally the beam inside the film will meet the front face again and a fraction is transmitted to form a beam parallel to the original reflected beam. See figure 20.

So we have 2 beams reflected from the film, 1 from the front surface and 1 from the rear. The optical path difference between these 2 components is:

$\Delta = n_f(AB + BC) - n_1 AD$

$AB = BC = d/\cos\theta_t$,

$AD = AC\sin\theta_i = AC(n_f/n_i)\sin\theta_t$ using Snell's law

and $AC = 2d\tan\theta_t$ so

$$\Delta = \frac{2n_f d}{\cos\theta_t}(1 - \sin^2\theta_t) = 2n_f d\cos\theta_t$$

If $n_1 = n_2$ then 1 beam suffers an internal reflection and the other beam an external reflection at a $n_1 : n_f$ interface which introduces a phase difference of π if there is no absorption (see Stokes treatment earlier). So the phase difference between the beams is:

$\delta = (2\pi/\lambda)2n_f d\cos\theta_t \pm \pi$

Hence for a maximum $d\cos\theta_t = (2m+1)\lambda_f/4$

where m is an integer and $\lambda_f = \lambda/n_f$.

Fringes of equal inclination are seen if d is fixed and θ_i is varied. Note the factor of $\lambda_f/4$, maxima and minima occur at multiples of quarter wavelengths. Filters and interference films are often called *quarter wave stacks* because of this.

Fringes of equal thickness are seen if the dn_f term dominates rather than the θ_i term. This will be the case if d is only $\approx$ few λ. Such fringes are very common:

- soap films - bubbles
- metal oxide films around welds
- oil films on puddles
- thin plastic films - cling film

The fringes are particularly strong when d is small and the illumination is near normal. Under such conditions $(m + 1/2)\lambda_o = 2n_f d_m$.

A simple set-up for observing the fringes is shown in figure 21.

30 FRINGES OF EQUAL THICKNESS IN A THIN WEDGE

It is easy to setup a thin wedge dielectric film between 2 glass slides. See figure 22.

In such a configuration the thickness is given by:

$d = \alpha x$

where α is the wedge angle and x is the distance from the apex. So we have fringes at positions x_m given by:

$(m + 1/2)\lambda_o = 2\alpha x_m n_f$ or

$x_m = (m + 1/2)\lambda_f/(2\alpha)$.

The fringe separation is:

$\Delta x = \lambda_f/(2\alpha)$

Such an arrangement can be used to measure α or λ.

31 MULTIPLE BEAM INTERFERENCE

As you may have noted in the analysis of interference from dielectric films above we ignored the presence of multiple reflections within the film. If the reflectivity at each interface is low this is a good approximation but if it is high it must not be ignored. See figure 23.

The optical path difference between successive reflections is as before:

$\Delta = 2n_f d \cos\theta_t$

but now we must also take account of the change in amplitude as the number of reflections increases.

We will use the same notation as that used for the Stokes' treatment.

r,t for the external amplitude coefficients

r',t' for the internal amplitude coefficients

remember $r = -r'$ and $tt' = 1 - r^2$

The relative amplitudes of the reflected beams are:

$r \;\; tr't' \;\; tr'^3t' \;\; tr'^5t' \; \ldots$

The relative amplitudes of the transmitted beams are:

$tt' \;\; tr'^2t' \;\; tr'^4t' \;\; tr'^6t' \; \ldots$

We find the total amplitude by vector addition of the amplitudes.

Suppose $\Delta = m\lambda$ so that all the reflected beams are in phase then the reflected amplitude is:

$$A_r = r - trt'(1 + r^2 + r^4 + \cdots) = r - trt'/(1 - r^2)$$

using the sum of a G.S. where $r^2 < 1$. Note the negative sign introduced by the difference in phase between internal and external reflections.

But $tt' = 1 - r^2$ so $A_r = 0$.

A phasor diagram, figure 24, indicates what is going on. All the reflected beams that exit from the film cancel the directly reflected beam.

We can look at the opposite case of $\Delta = (m + 1/2)\lambda$. Then alternate reflected beams are out of phase.

$$A_r = r + trt' - tr^3t' + tr^5t' + \cdots = r + rtt'(1 - r^2 + r^4 - r^6 + \cdots)$$

Again we can sum the G.S. and use $tt' = 1 - r^2$ to give:

$$A_r = 2r/(1 + r^2)$$

Again a phasor diagram, figure 24, illustrates what is happening.

In general the phase difference between successive reflections is:

$$\delta = k_o\Delta$$

Now the complex amplitude sum has the form:

$$A_r = \exp(i\omega t)(r + r'tt'\exp(i\delta)(1 + r'^2\exp(-i\delta) + \cdots + (r'^2\exp(-i\delta))^{N-2}))$$

If $r'^2\exp(-i\delta) < 1$ then the geometric series converges giving

$$A_r = \exp(i\omega t)(r + \frac{r'tt'\exp(-i\delta)}{1 - r'^2\exp(-i\delta)}) = \exp(i\omega t)(r\frac{1 - \exp(-i\delta)}{1 - r^2\exp(-i\delta)}$$

The intensity of a beam is proportional to the square of the amplitude. Since we have assumed that the incident amplitude is 1 the ratio of the total reflected intensity to the incident intensity is given by:

$$R = A_r A_r^* = 2r^2\frac{1 - \cos\delta}{1 + r^4 - 2r^2\cos\delta}$$

Similarly we can do the same analysis for the transmitted beam. A similar G.P. results and we get a ratio of the transmitted intensity to the incident intensity:

$$T = \frac{1}{1 + (2r/(1 - r^2))^2\sin^2\delta/2}$$

Note that $R+T = 1$ because we have assumed there are NO absorption losses. Maximum transmission occurs when $\delta = 2\pi m$ and then reflection is at a minimum.

The phasor diagram for the general case has a spiral form and is showm in figure 25. As δ changes so the resultant changes.

We can plot T (or R) as a function of δ. The sharpness of the peaks at multiples of 2π depends on the amplitude reflection coefficient r. The larger r the sharper the peaks.

It is customary to define a *coefficient of finesse*:

$$F = (\frac{2r}{1 - r^2})^2$$

Then substituting for F we get:

$$R = \frac{F\sin^2\delta/2}{1 + F\sin^2\delta/2}$$

$$T = \frac{1}{1 + F\sin^2\delta/2}$$

The last function is called the *Airy Function*. See figure 26.

Note that the fringes are no longer cosinusoidal as in the 2 beam interference case. The larger the number of effective beams (large r) the sharper the fringes.

The film acts as a resonant cavity. Standing waves are set up between the reflecting interfaces. The positions of the fringe peaks correspond to an exact number of half wavelengths fitting between the film boundaries.

32 THE FABRY-PEROT INTERFEROMETER

A very simple multi-beam interferometer can be constructed using 2 parallel reflecting plates. Such a device was first produced by Charles Fabry and Alfred Perot in the 1890's. It is illustrated in figure 27.

The 2 plates constitute an *etalon*. The modern equivalent is a resonant cavity found in many LASER systems.

If the plates are adjusted to be parallel then *sharp fringes of equal inclination* are seen. The phase difference between successive beams is:

$\delta = (4\pi/\lambda)n_f d\cos\theta_t + \phi$

where ϕ is the difference in phase introduced by reflection at the thin metallic films. The maxima are seen at θ_t given by:

$m\lambda_o = 2n_f d\cos\theta_t + \phi\lambda_o/\pi$

In practice the last term is unimportant and introduces a small shift of the fringe pattern.

If the coefficient of finesse (defined above) is large, $F > 30$, then the fringes are very sharp and the etalon can be used as a high resolution spectrometer. If d, n_f and ϕ are fixed then θ_t varies with λ_o. Two very close wavelengths are resolved as two sets of separated ring systems. Try and estimate the resolving power $\lambda/\delta\lambda$ for $F = 30$ say.

33 INTERFERENCE FILTERS

The phenomenon of multiple beam interference in thin films can be employed to construct very narrow waveband transmission filters and anti-reflection coatings.

The thickness d is made small so that it dominates the expression for the phase change δ. The transmission curve has the form of Airy's function. For maximum:

$d \cos \theta_t \approx (2m + 1)\lambda_f / 4$

Therefore such filters are constructed using $\lambda/4$ layers. They are used in:

- blooming on camera lens
- UV filters
- X-ray optics - multilayer reflecting coatings working at normal incidence

34 FRAUNHOFER DIFFRACTION

Consider the shadow cast by a slit on a screen from a monochromatic point source. If the screen is close to the slit the shadow looks very much like the slit except for some slight blurring at the edges. As we increase the distance of the screen from the slit the shadow becomes blurred and eventually forms a pattern which bears little resemblence to the shape of the aperture.

If we now gradually shorten the wavelength of the light the shadow will regain the outline of the slit.

In such an experiment we are moving from the regime of *near field diffraction* to that of *far field diffraction*. The pattern seen depends on the ratio of the dimensions of the system to the wavelength of the light.

Fraunhofer diffraction is the limiting case of far field diffraction. It concerns the diffraction of *plane wavefronts* and small diffraction angles. This is the approximation we used in the analysis of Young's slits. Fraunhofer diffraction is a generalisation of the Young's slits experiment.

Lenses can be used to transform spherical wavefronts into plane wavefronts and vice versa. See figure 28a.

Spherical wavefronts from a point, monochromatic source are converted to plane wavefronts by a *collimating* lens, L_1. These plane wavefronts are diffracted by an aperture plane Σ. The diffracted waves can be resolved (described by) a set of plane wave components propogating at angles θ to the axis. Each of these components is focused by a second, telescope lens, L_2. i.e. the diffracted planewaves are converted to converging spherical waves by the second lens and a diffraction pattern is formed by interference at the focal plane of L_2.

35 DIFFRACTION GRATINGS

Diffraction gratings are usually operated in the Fraunhofer diffraction domain.

Consider a series of N narrow, parallel, equally spaced slits. If these are introduced into a Fraunhofer diffraction setup the slits act as coherent line sources. See figure 31.

The path difference between adjacent slits at an angle θ is $\Delta = d\sin\theta$ so the phase difference is:

$\delta = kd\sin\theta$

If the amplitude from each slit is S then ignoring any obliquity factor the total amplitude at angle θ is:

$A(\theta) = S(1 + \exp(i\delta) + exp(i2\delta) + \cdots + \exp(i(N-1)\delta))$

Summing the G.S. gives:

$$A(\theta) = S\frac{1 - \exp(iN\delta)}{1 - \exp(i\delta)}$$

So the intensity (amplitude squared) at angle θ is:

$$I = AA^* = S^2\frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}$$

This is illustrated in figure 32 for the case of $N = 6$.

Principle maxima occur at $\delta/2 = 0, \pm\pi, \pm2\pi, \ldots$

Minima occur at $\delta/2 = \pm\pi/N, \pm 2\pi/N, \ldots$

Secondary maxima occur at $\delta/2 = \pm 3\pi/2N, \pm 5\pi/2N, \ldots$

If N is large only the principal maxima are visible. These correspond to the *orders of diffraction.*

The mth order is given by $m\lambda = d \sin\theta$

In the above we have not considered the effect of the width of each of the slits. This can be done using the Fourier integral formula for the Fraunhofer diffraction pattern.

When mounted in a Fraunhofer setup a grating with large N generates very sharp diffraction peaks for each wavelength present in the source. Therefore gratings are widely used as the dispersion elements in spectrometers from infra-red through to soft X-rays.

In practice gratings are constructed by scoring or etching grooves in a glass substrate. In modern gratings the grooves can be optimized in shape or *blazed* to maximize the intensity that is diffracted into certain orders for a chosen range of wavelengths.

36 GENERAL FORMULA FOR FRAUNHOFER DIFFRACTION

The diffraction occurs at plane Σ. At each point over this plane there is a complex amplitude:

$E = E_{yz} \exp i(\omega t - kx)$

By Huygen's principle this radiates wavelets over all angles. Consider the direction (θ, ϕ) as illustrated in figure 28b. We can calculate the optical path difference of the wavelets from the position (y, z) at angle (θ, ϕ) with respect to the origin:

$\Delta = y \sin\theta + z \sin\phi$ if the angles are small.

The phase difference wrt the origin is therefore:

$\delta = k(y \sin\theta + z \sin\phi)$

So the amplitude in the direction (θ, ϕ) from the area element $dzdy$ at position (y, z) is:

$\frac{E_{yz}}{x} \exp i(\omega t - kx + k(y \sin\theta + z \sin\phi))dydz$

We have assumed that the amplitude is proportional to the area. The factor $1/x$ comes from the inverse square law for the wavelets and we have ignored any *obliquity factor*. This is only an approximate expression for *small angles*.

To find the total amplitude at angle (θ, ϕ) we must perform a complex sum over all points on Σ. We do this by integration over all points for which $E_{yz} \neq 0$.

In the simple case $E_{yz} = E_0$ for a hole in Σ and $E_{yz} = 0$ otherwise. So we have:

$E(\theta, \phi) = E_0 \exp i(\omega t - kx) \int \int_{holes} \exp i(ky \sin\theta) \exp i(kz \sin\phi) dydz$

It is this amplitude which is focused by lens L_2 to a position on the screen given by:

$y' = f_2 \tan\theta \approx f_2\theta$

$z' = f_2 \tan\phi \approx f_2\phi$

where we can make the tangent approximation if the angles are small. Remember the lens converts plane wavefronts to spherical wavefronts.

The integral above has the form of a *2-D Fourier Transform*. The interference pattern on the screen is the Fourier transform of the distribution of holes over Σ. Refer to the earlier mathematics notes on the Fourier Transform.

This is a general result under the *plane wave, small angle* approximation. The Fraunhofer diffraction pattern is the Fourier Transform of the diffracting object. In general the object can modulate the amplitude and/or phase of the incident radiation. The E_{yz} term may need to be inside the integral.

The detected distribution is given by the intensity (amplitude suared):

$I = E(y', z')E^*(y', z')$ which is the square of the transform, i.e. the power.

Examples of Fraunhofer diffraction patterns are shown in figure 29.

37 RECTANGULAR APERTURES

For a rectangular aperture we can factorize E_{yz}: $E_{yz} = E_0 S_y S_z$ where

$S_y = 1$ for $-a/2 < y < a/2$ and is zero elsewhere

$S_z = 1$ for $-b/2 < z < b/2$ and is zero elsewhere

a and b are the dimensions of the aperture. We make the following substitutions to simplify the equations:

$\alpha = \frac{ka\sin\theta}{2} \approx \frac{ka\theta}{2}$

$\beta = \frac{kb\sin\phi}{2} \approx \frac{kb\phi}{2}$

Note these substitutions are just a rescaling of the diffraction angles in the small angle approximation when $k = constant$.

The 2-D Fourier Integrals are separated in this case and the result is:

$E(\alpha,\beta) = \frac{E_0}{x}\exp i(\omega t - kx) a\frac{\sin\alpha}{\alpha} b\frac{\sin\beta}{\beta}$

The intensity is given by:

$I = EE^* = \frac{E_0^2}{x^2} a^2 b^2 \frac{\sin^2\alpha}{\alpha^2}\frac{\sin^2\beta}{\beta^2}$

There are zeros at $\alpha = \pm m\pi$ and $\beta = \pm l\pi$ where m and l are integers.

The separation of the minima on the screen is:

$\Delta y' = \frac{f_2\lambda}{a}$ and $\Delta z' = \frac{f_2\lambda}{b}$

38 YOUNG'S SLITS REVISITED

When we dealt with wavefront splitting 2 source interference we ignored the width and height of the sources (slits). We can use the general formula to calculate the true Fraunhofer diffraction pattern of such a setup.

The aperture function is seperable:

$$S(y,z) = S_y(y)S_z(z)$$

where $S_z(z) = 1$ for $-b/2 < z < +b/2$ and $S_y(y) = 1$ for $(-d-a)/2 < y < (-d+a)/2$ and $(d-a)/2 < y < (d+a)/2$. The functions are $= 0$ elsewhere.

The z integration gives:

$$b\frac{\sin\beta}{\beta}$$

where

$$\beta = \frac{kb\phi}{2}$$

The y integral splits into 2 parts

$$\int_{\frac{-d-a}{2}}^{\frac{-d+a}{2}} \exp(iky\theta)dy + \int_{\frac{d-a}{2}}^{\frac{d+a}{2}} \exp(iky\theta)dy$$

performing the integration and substituting the limits gives us:

$$\frac{1}{ik\theta}(\exp(\frac{-ik\theta d}{2})\exp(\frac{+ik\theta a}{2}) - \exp(\frac{-ik\theta d}{2})\exp(\frac{-ik\theta a}{2}) + \exp(\frac{+ik\theta d}{2})\exp(\frac{+ik\theta a}{2}) - \exp(\frac{+ik\theta d}{2})\exp(\frac{-ik\theta a}{2}$$

$$= \frac{2a}{k\theta a/2}\sin\frac{k\theta a}{2}\cos\frac{k\theta d}{2}$$

If we let

$$\delta = k\theta d$$

$$\alpha = \frac{k\theta a}{2}$$

then y integral is

$$= 2a\frac{\sin\alpha}{\alpha}\cos\frac{\delta}{2}$$

This is plotted in figure 34 with the corresponding intensity in the bottom panel. The interference fringes are the last term. The finite width of the slits introduces a modulation of these fringes in the form of a sinc function.

So the complete 2-D Fraunhofer amplitude of Young's slits is given by:

$$\frac{E_o}{x} \exp i(\omega t - kx) b \frac{\sin\beta}{\beta} 2a \frac{\sin\alpha}{\alpha} \cos\frac{\delta}{2}$$

39 CIRCULAR APERTURES

We now have an aperture function:

$E_{yz} = E_0 S(y,z)$ where $S(y,z) = 1$ for $\sqrt{y^2+z^2} = r \leq a$ and $S(y,z) = 0$ for $r > a$.

The Fourier integrals are best rewritten in polar coordinates. The result involves a Bessel function of the first kind. The Bessel function of order m is given by:

$J_m(u) = \frac{i^{-m}}{2\pi} \int_0^{2\pi} \exp i(mv + u\cos v) dv$

Bessel functions have series expansions just like the circular functions sine and cosine. They have a slowly decreasing oscillating profile.

The complex amplitude of the diffraction pattern is:

$E(r') = \frac{E_0}{x} \exp i(\omega t - kx) 2\pi a^2 \frac{J_1(kar'/f_2)}{kar'/f_2}$

where r' is the radial position on the focal plane of the second lens. Thus the intensity is given by:

$I(r') = E(r')E(r')^* = \frac{E_0^2}{x^2}(2\pi a^2)^2 \frac{J_1^2(kar'/f_2)}{(kar'/f_2)^2}$

This is known as the Airy disk after the Astonomer Royal Sir George Biddel Airy (1801-92) who first derived this equation. This function is illustrated in figure 30.

The first minimum (zero) in intensity occurs at:

$\frac{kar'}{f_2} = 3.83$

so

$r' = \frac{3.83 f_2}{ka} = 1.22 f_2 \frac{\lambda}{2a}$

The peak of the first secondary maximum is only 0.0175 the peak intensity; the ring pattern is in fact very faint.

40 DISH ANTENNAE - PARABOLIC MIRRORS

At high radio frequencies (small wavelengths of cms) it is common to use a parabolic dish antenna. For a transmitting antenna the signal is fed to the focus of the parabola using a waveguide terminated by a horn.

The horn feed irradiates the dish with spherical waves which reflect off the parabola and are converted into plane waves.

For reception the opposite occurs. Plane waves are reflected and converted to spherical waves which converge on the collecting horn. See figure 33.

The sum of the amplitudes from an incident beam at an angle θ is identical to the Fraunhofer diffraction pattern for the aperture. i.e. the angular response of the dish in either Tx or Rx is the Airy disk. So the beam angle is approximated by the angle of the first minimum:

$$\theta_B = \frac{1.22\lambda}{D}$$

We define the antenna gain as:

$$G = \frac{I_{peak}}{I_{mean}}$$

i.e. we imagine all the radiated power to be spread over all angles, 4π steradians, and compare the resulting flux with the peak flux obtained in the beam. This is approximately:

$$G = \frac{4\pi}{Area_{beam}}$$

(Note for a simple dipole $G \approx 2 \equiv 3dB$.)

For a parabolic dish the angular area of the beam is $A_B \approx \pi\theta_B^2$ if the beam is small so:

$$G \approx \frac{4\pi}{\pi\theta_B^2} = \frac{4}{\theta_B^2}$$

provided $\lambda/D \ll 1$

The parabolic primary mirror (or the objective lens) of an optical telescope works in the same way. Therefore the angular resolution of a telescope is:

$$\theta_R = \frac{1.22\lambda}{D}$$

Where D is the diameter of the primary aperture. See figure 33.

This assumes the *Rayleigh criterion* that 2 point sources will be resolved providing their angular separation is greater than that corresponding to the peak of 1 being on the first diffraction minimum of the other.